OPTICS

PHYS 311

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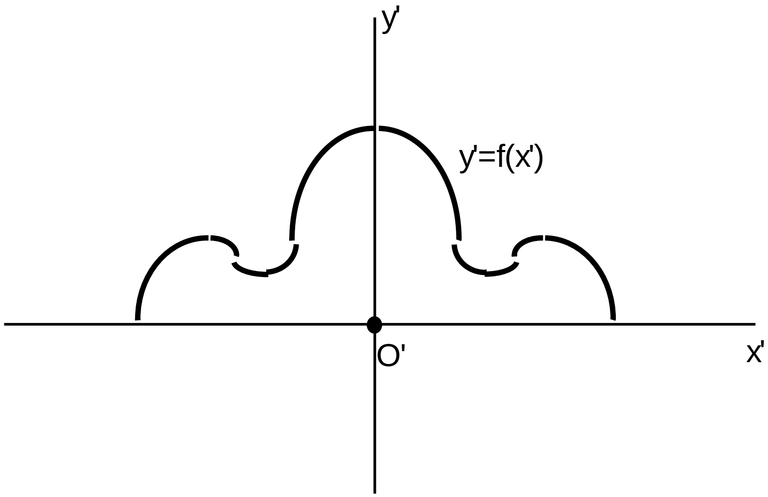
WAVE EQUATION

Introduction

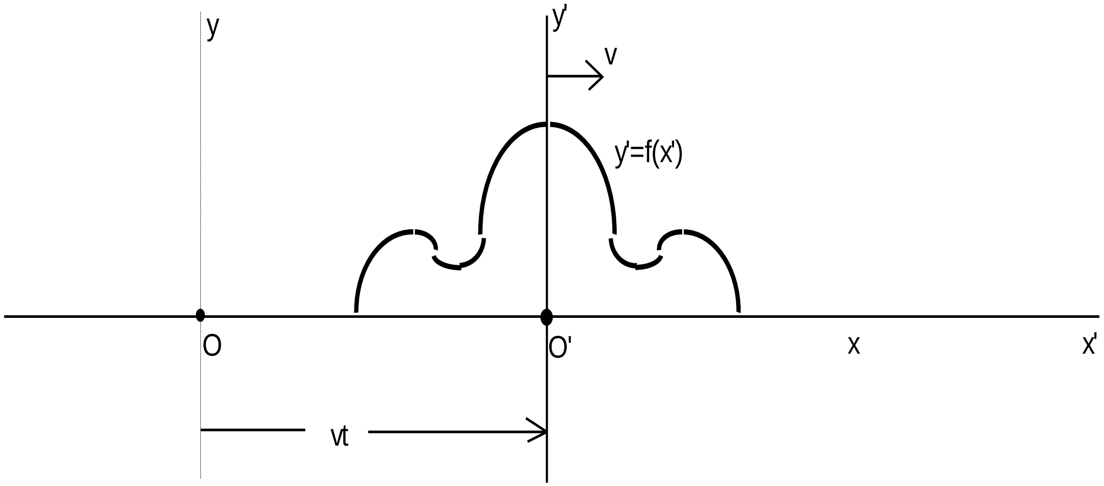
Before we can understand how light moves from one medium to another and how it interacts with lenses and mirrors, we must be able to describe its motion mathematically

1. The most general form of a traveling wave.
2. The differential equation it satisfies.



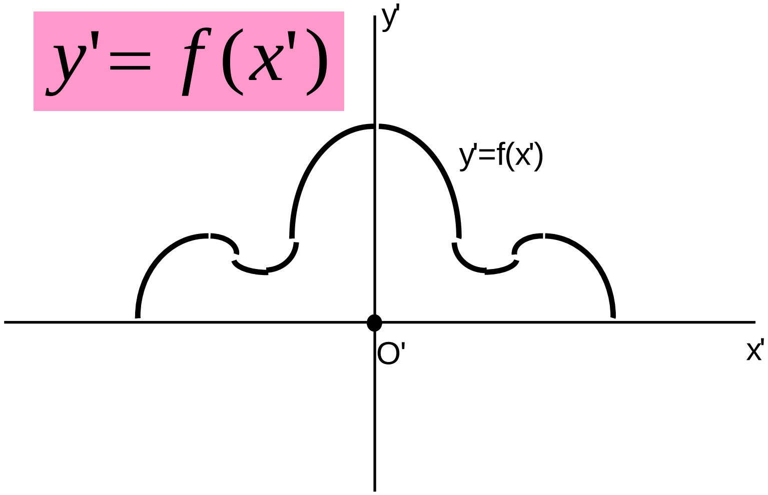
One dimensional wave pulse of arbitrary shape fixed to a coordinate system O'(x',y')

The pulse maintains its shape



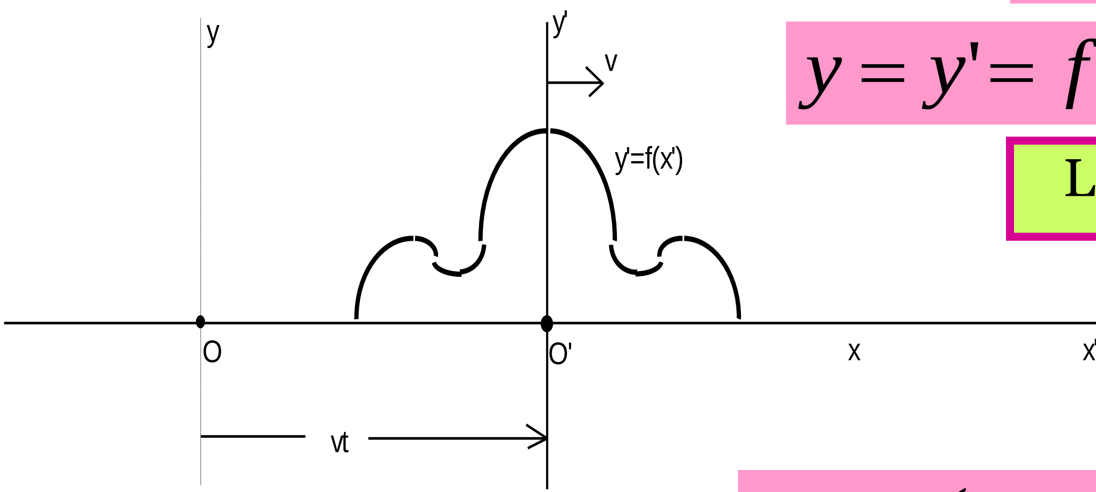
The O' system & the pulse move to the right along the x-axis at a uniform speed v relative to a fixed coordinate system O(x,y)

$$y' = f(x')$$



Any point P on the pulse can be described by either of the two coordinates x or x'

$$x' = x - \upsilon t$$



$$y = y' = f(x') = f(x - \upsilon t)$$

Left movement→ +υ

$$y = f(x \pm \upsilon t)$$

$$\psi = \psi(x, y, z, t) = f(\vec{r} \pm \vec{\upsilon} t)$$

The general form of a traveling wave

$$\psi(\vec{r}, t)\big|_{t=0} = \psi(\vec{r})$$

Shape or Profile

One Dimensional Waves

$$\psi(\vec{r},t) \rightarrow \psi(x,t) = \psi(x') \quad \Leftarrow \quad x' = x - \upsilon t$$

The variation of ψ with respect to position is given by:

The variation of ψ with respect to time is given by:

$$\frac{\partial \psi(x,t)}{\partial x} = \frac{\partial x'}{\partial x}\frac{d\psi(x')}{dx'} = \frac{d\psi(x')}{dx'}$$

$$\frac{\partial \psi(x,t)}{\partial t} = \frac{\partial x'}{\partial t}\frac{d\psi(x')}{dx'} = \pm\upsilon\frac{d\psi(x')}{dx'}$$

Taking the second derivative of these yields:

One Dimensional Waves

Taking the second derivative of these yields:

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\psi(x,t)}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\psi(x')}{\partial x'}\right)$$

$$= \left(\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}\right)\frac{d\psi(x')}{dx'}$$

$$= \frac{d^2\psi(x')}{dx'^2}$$

$$\frac{\partial^2\psi(x,t)}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial\psi(x,t)}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}\left(\pm\upsilon\frac{d\psi(x')}{dx'}\right)$$

$$= \left(\pm\upsilon\frac{\partial}{\partial x'}\right)\left(\pm\upsilon\frac{d\psi(x')}{dx'}\right)$$

$$= \upsilon^2\frac{d^2\psi(x')}{dx'^2}$$

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{1}{\upsilon^2}\frac{\partial^2\psi(x,t)}{\partial t^2}$$

THE WAVE EQUATION

Three Dimensional Waves

Extending to three dimensions

$$\partial/\partial x \;\Rightarrow\; \vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Directional derivative
≡ gradient

THE WAVE EQUATION

$$\nabla^2 \psi(\vec{r},t) = \frac{1}{\upsilon^2}\frac{\partial^2 \psi(\vec{r},t)}{\partial t^2}$$

∇^2 is called the Laplacian operator

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

CARTESIAN COORDINATES

Harmonic Waves

Of special importance are simple harmonic waves that involve the sine or cosine functions

$$\psi(\vec{x},t) = A\, sin[\, \vec{k}\cdot\left(\vec{x} \pm \vec{\upsilon}t\right) + \phi_o \,]$$

A — **Amplitude**

k — **Propagation number**

ϕ_0 — **Initial phase**
Epoch angle

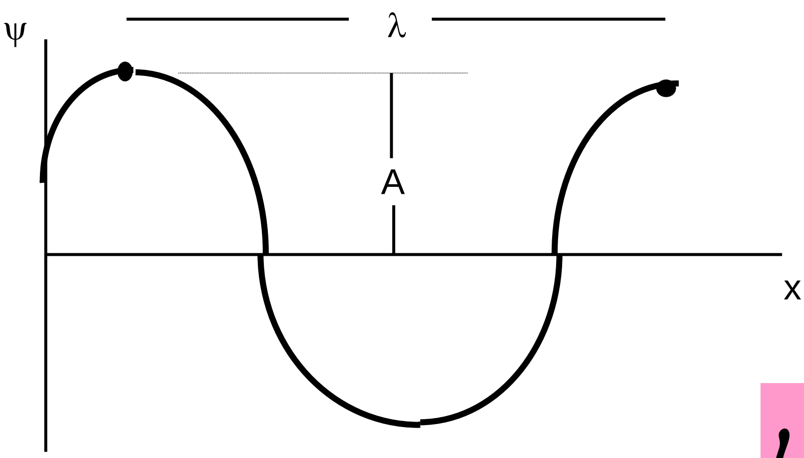
Periodic waves

Fourier Series → A linear combination ≡ Periodic wave

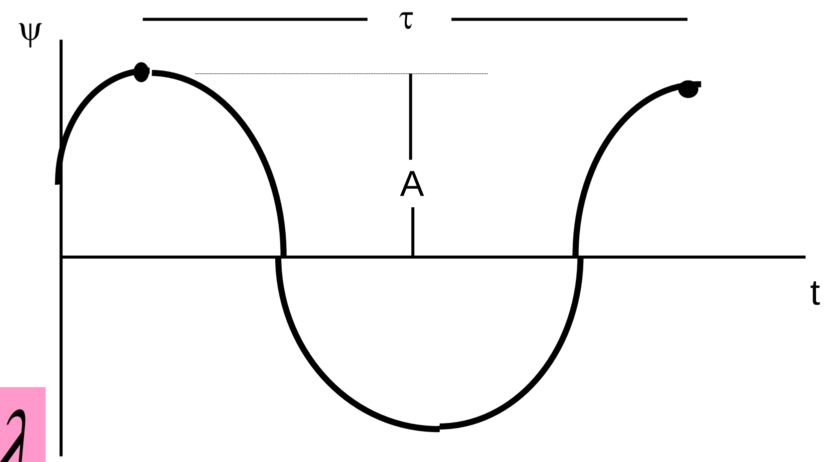
Harmonic Waves

What is the physical interpretation of HW equation?

$$\psi(\vec{x},t) = A\sin[\vec{k}\cdot(\vec{x} \pm \vec{\upsilon}t) + \varepsilon]$$



t = constant



x = constant

$$\upsilon = \nu\lambda$$

λ

wavelength

$$k = 2\pi/\lambda$$

$$\kappa = 1/\lambda$$

Wave number

$$\omega = 2\pi\nu$$

Angular frequency

τ

Period

$$\tau = 1/\nu$$

Harmonic Waves

$$\psi(\vec{x},t) = A\, sin[\,\vec{k}\cdot(\vec{x} \pm \vec{\upsilon}t) + \phi_0\,]$$

$$\varphi = \vec{k}\cdot(\vec{x} \pm \vec{\upsilon}t) + \phi_0$$

Phase

When *x* and *t* change together in such a way that φ is constant, the displacement $\psi = A\ sin\ \varphi$ is also a constant.

Describes the motion of a fixed point on the wave form

$$d\varphi = 0 = \vec{k}\cdot(d\vec{x} \pm \vec{\upsilon}dt)$$

$$\frac{d\vec{x}}{dt} = \pm\vec{\upsilon}$$

Wave velocity
Phase velocity

ϕ_0

Initial phase

when $x = 0$ & $t = 0$

$$\psi_o = A\, sin\, \phi_0$$

Complex Representation of Waves

It is convenient to represent wavefunctions as complex functions

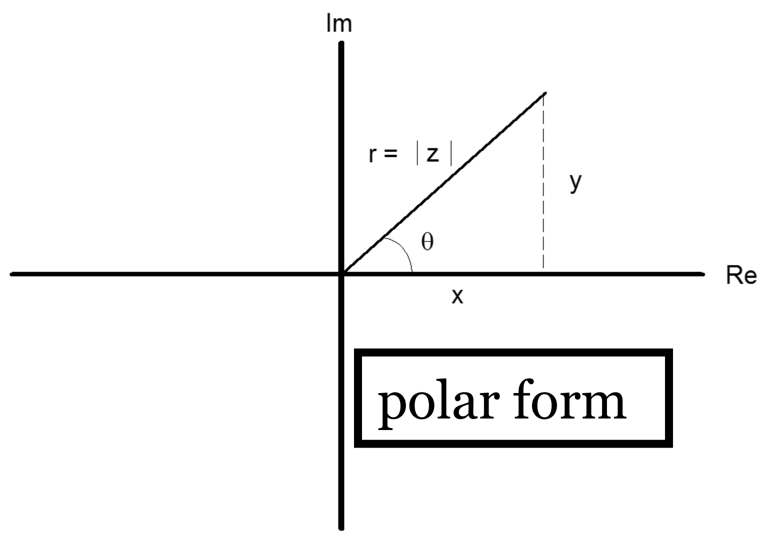
$$\psi(\vec{x},t) = A\, sin[\,\vec{k}\cdot(\vec{x} \pm \vec{\upsilon}t) + \phi_0\,]$$

$$\psi(x,t) = Ae^{i(\vec{k}\cdot\vec{x} - \omega t + \phi_0)}$$

mathematically simpler to use

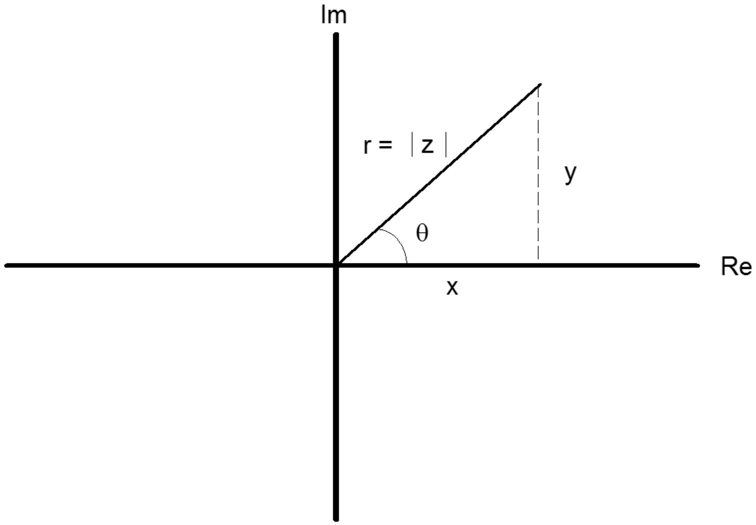
$$z = x + iy$$

Complex number



The **magnitude** of z, symbolized by r, is called **absolute value** or **modulus**

Complex Representation of Waves



$$z = x + iy \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ r^2 = x^2 + y^2 \end{cases}$$

$$z = r(\cos\theta + i\sin\theta)$$

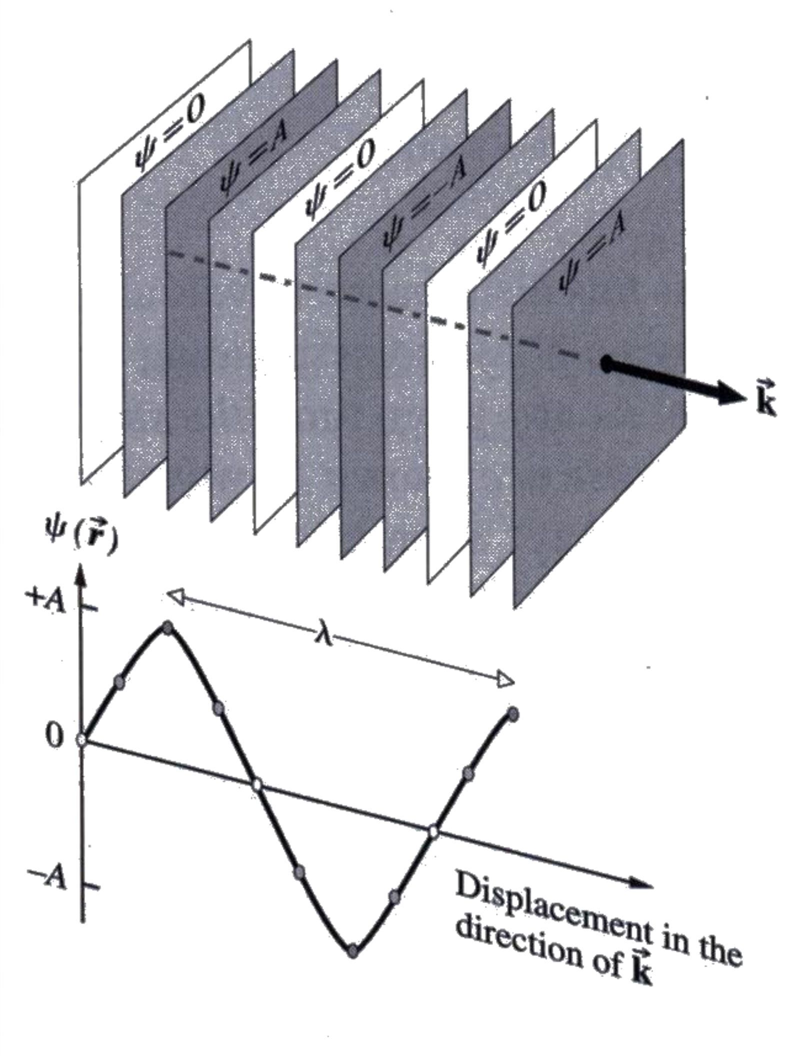
Complex conjugate

$z^* = x - iy$ or $z^* = re^{-i\theta}$

Euler's formula $\rightarrow$ $z = re^{i\theta}$ $\quad \theta = \tan^{-1}\dfrac{y}{x}$

$$\psi(x,t) = Ae^{i(\vec{k}\cdot\vec{x}-\omega t+\varepsilon)} \quad \begin{cases} \mathrm{Re}(\psi) = A\cos\left(\vec{k}\cdot\vec{x}-\omega t+\varepsilon\right) \\ \mathrm{Im}(\psi) = A\sin\left(\vec{k}\cdot\vec{x}-\omega t+\varepsilon\right) \end{cases}$$

Plane Waves



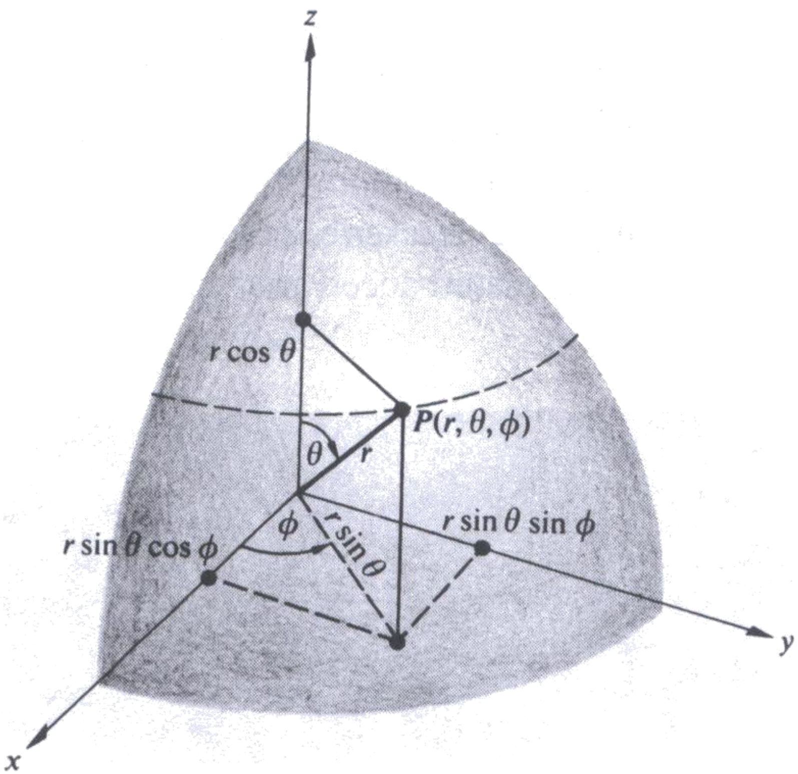
These planes are called the **wavefronts** of the disturbance

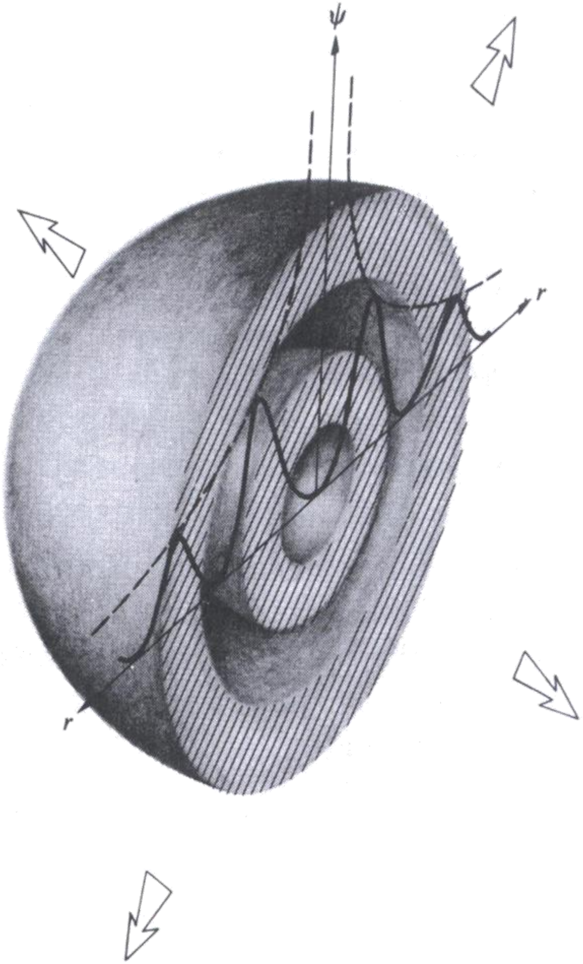
$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

A wave which satisfies the above wave equation is called a **PLANE WAVE**

$$\psi = Ae^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}$$

Spherical Waves

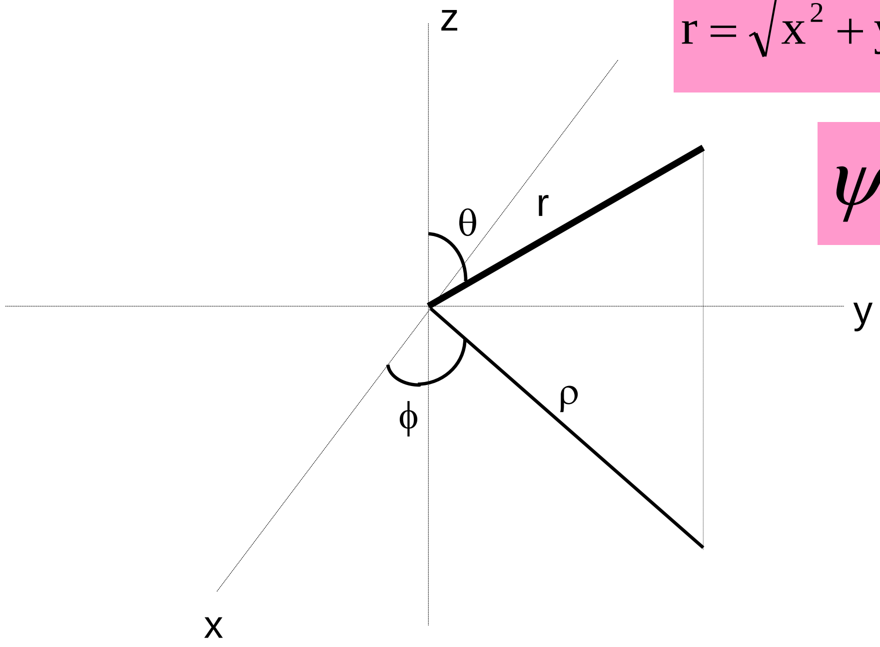




Spherical Waves

In this coordinate system, the Laplacian operator becomes

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$



$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \qquad \phi = tan^{-1}\frac{y}{x}$$

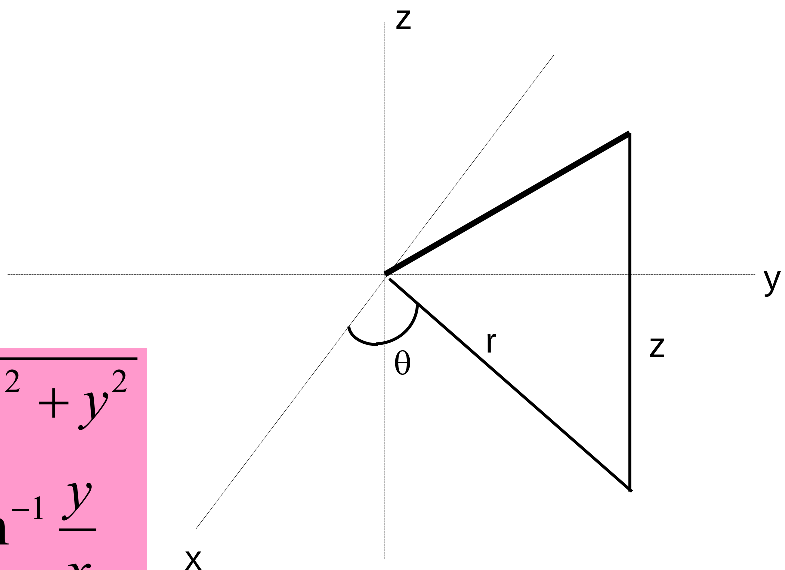
$$\psi(\vec{r}) = \psi(r,\theta,\phi) = \psi(r)$$

$$\psi(r) = \frac{A}{r}e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}$$

spherical wavefunction

the amplitude of a spherical wave decreases as it moves away from its source

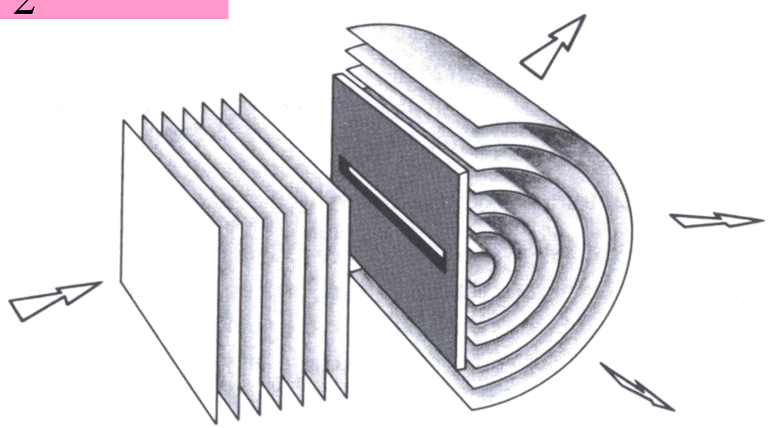
Cylindrical Waves



$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$z = z$$



In this coordinate system, the Laplacian operator becomes

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

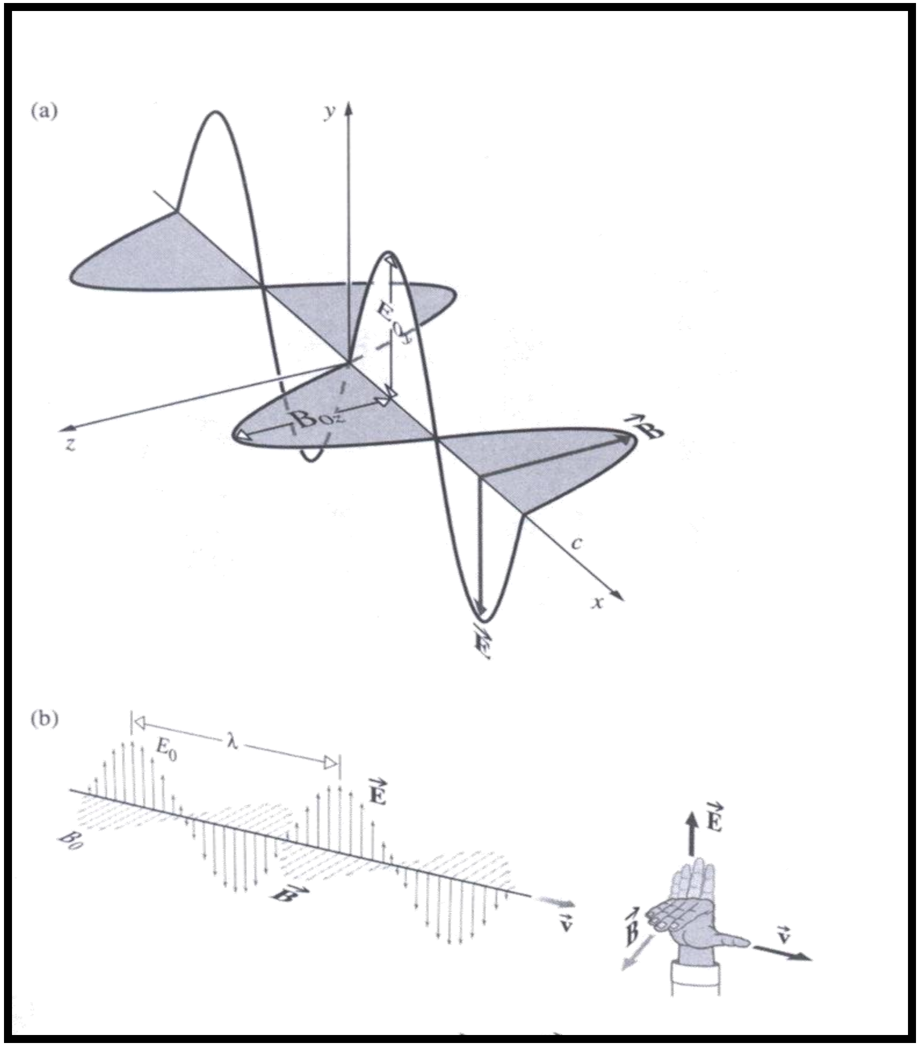
The requirement of cylindrical symmetry means that

$$\psi(\vec{r}) = \psi(r,\theta,z) = \psi(r)$$

$$\psi(r,t) \approx \frac{A}{\sqrt{\rho}} e^{i(k\rho \pm \omega t)}$$

cylindrical wavefunction

ELECTROMAGNETIC WAVES



$$E = E_o \, sin(\vec{k}.\vec{r} - \omega t)$$

$$B = B_o \, sin(\vec{k}.\vec{r} - \omega t)$$

$$E = cB$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

ENERGY IN AN EM WAVE

As with any wave, the EM wave transports energy

Energy density stored in an E field

$$u_E = \frac{1}{2}\varepsilon_0 E^2$$

Energy density stored in a B field

$$u_B = \frac{1}{2\mu_0} B^2$$

$$E = cB$$

$$c = 1\Big/\sqrt{\mu_0 \varepsilon_0}$$

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 c^2 B^2 = \frac{1}{2}\frac{\varepsilon_0}{\mu_0 \varepsilon_0} B^2 = \frac{1}{2\mu_0} B^2 = u_B$$

The total energy density is shared between the constituent electric and magnetic fields

$$u = u_E + u_B$$

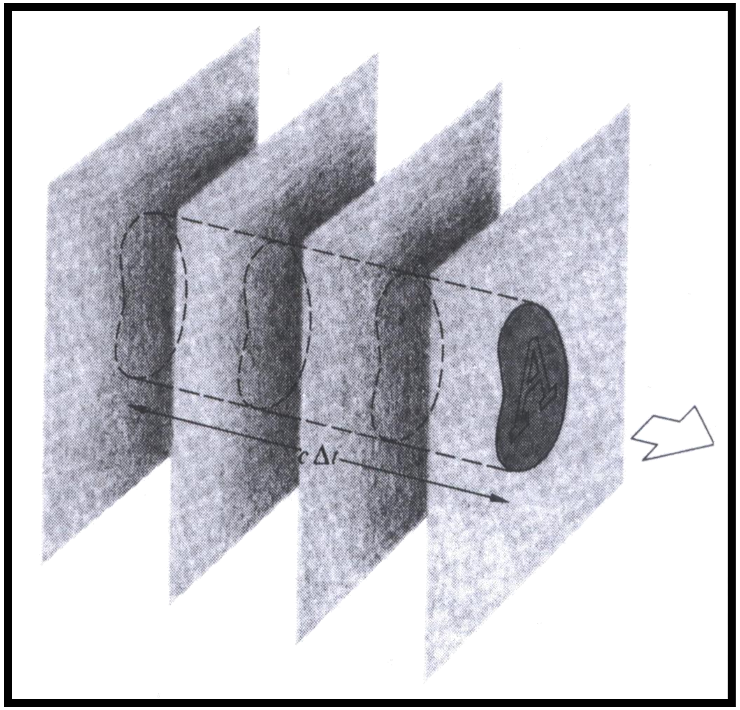
$$u = \varepsilon_0 E^2 = \frac{1}{\mu_o} B^2$$

ENERGY IN AN EM WAVE

What is the EM energy flow associated with a traveling wave?

S: transport of energy per unit time across a unit area (W/m^2)

During a very small interval Δt, only the energy contained in the cylindrical volume, $u(c\,\Delta t A)$, will cross A



$$S = \frac{uc\Delta tA}{\Delta tA} = uc = \frac{1}{\mu_o} EB$$

For isotropic media, the energy flows in the direction of the propagation of the wave:

$$\mathbf{S} = \frac{1}{\mu_o}\mathbf{E}\times\mathbf{B} = c^2\varepsilon_o\mathbf{E}\times\mathbf{B}$$

Poynting vector

IRRADIANCE

Irradiance: *AVERAGE* energy per unit area per unit time

$$\mathrm{I} \equiv \left\langle \left|\mathbf{S}\right| \right\rangle_{\mathrm{T}} = \mathrm{c}^2 \varepsilon_{\mathrm{o}} \left| \mathbf{E_o} \times \mathbf{B_o} \right| \left\langle sin^2 (\mathbf{k.r} \pm \omega \mathrm{t}) \right\rangle$$

$$I \equiv \left\langle S \right\rangle_T = \frac{c\varepsilon_o}{2} E_o^2 \Rightarrow I = c\varepsilon_o \left\langle E^2 \right\rangle_T = \frac{c}{\mu_o} \left\langle B^2 \right\rangle_T$$

E is considerably more effective at exerting forces and doing work on charges than **B**

E is called the **OPTICAL FIELD**

LIGHT POLARIZATION

The direction of E is known as the polarization of the wave.

$$\vec{E} = E_o \, sin(kz - \omega t) \, \widehat{x}$$

$$\vec{B} = B_o \, sin(kz - \omega t) \, \widehat{y}$$

$$\vec{S} = \varepsilon_0 c E_0^2 \, sin^2(kz - \omega t) \, \widehat{z}$$

E is called the **OPTICAL FIELD**

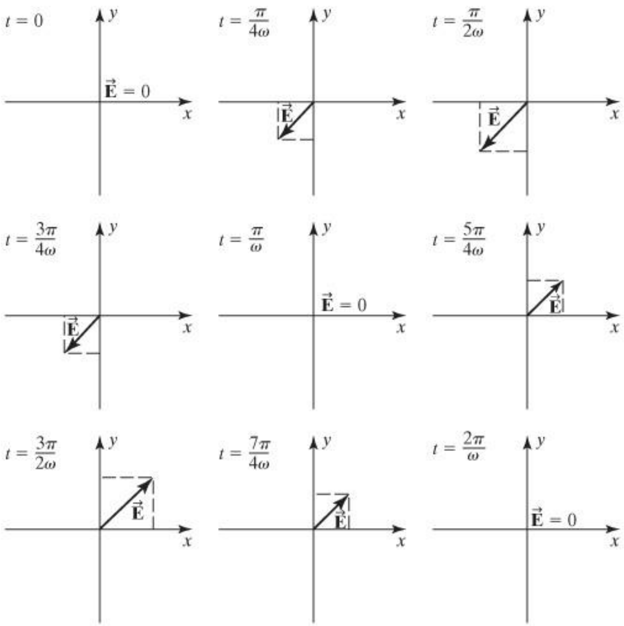
The polaraization of an EM wave determines the direction of the force that the EM wave exerts on a charged particle in the path of the wave.
← Lorentz force law

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$

LIGHT POLARIZATION

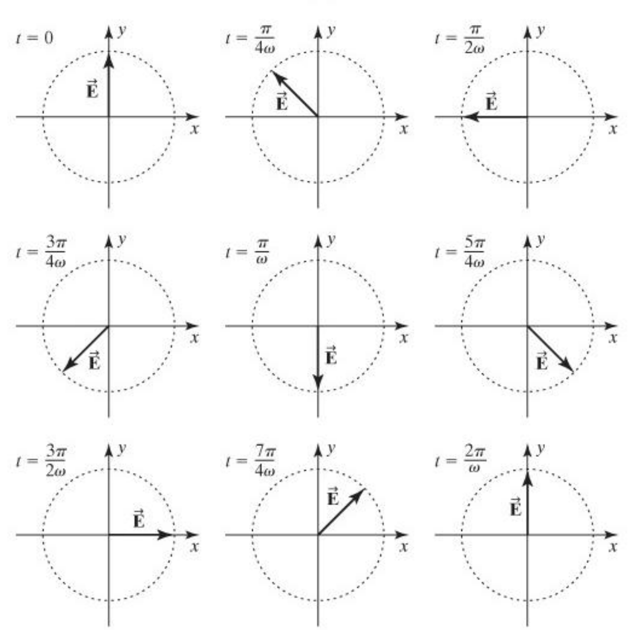
Linear polarization

circular polarization



$\vec{\mathbf{E}} = E_0 \sin(kz - \omega t)\hat{\boldsymbol{x}} + E_0 \sin(kz - \omega t)\hat{\boldsymbol{y}}$

(a)



$\vec{\mathbf{E}} = E_0 \sin(kz - \omega t)\hat{\boldsymbol{x}} + E_0 \cos(kz - \omega t)\hat{\boldsymbol{y}} = E_0 \sin(kz - \omega t)\hat{\boldsymbol{x}} + E_0 \sin(kz - \omega t + \pi/2)\hat{\boldsymbol{y}}$

(b)

Random polarized, partial polarized light
← How to convert them to polarized light???

DOPPLER EFFECT

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1-\frac{\upsilon}{c}}{1+\frac{\upsilon}{c}}}$$

υ is the relative velocity between the source and observer

υ + ← approaching each other
υ - ← separating from each other